Tutorial 7

In the following problems, V is a finite-dimensional vector space over a field $\mathbb{F}$ unless otherwise stated.

1. Suppose V is an inner product space and $T \in \mathcal{L}(V)$ is self-adjoint. What can you say about the generalized eigenspaces of T?

2. Let $\mathbb{F} = \mathbb{C}$. Suppose $T \in \mathcal{L}(V)$ and $c \in \mathbb{C}$ are such that $T^2 - c^2 I$ is nilpotent. What can you conclude about c?

3. (8.A.3) Suppose $T \in \mathcal{L}(V)$ is invertible. Show that for all $c \in \mathbb{F}$ nonzero,

$$G(c, T) = G\left(\tfrac{1}{c}, T^{-1}\right)$$

4. (8A.9) Suppose $S, T \in \mathcal{L}(V)$ are such that ST is nilpotent. Prove that TS is nilpotent.

5. Suppose $S, T \in \mathcal{L}(V)$ are commuting nilpotent operators. Show that $S+T$ is nilpotent.

6. Suppose $S, T \in \mathcal{L}(V)$ are nilpotent and that there exists $u, v \in V$ such that $S^{n-1}(u)$ and $T^{n-1}(v)$ are nonzero, where $n = \dim V$. Show that S and T are similar.

7. Let $V = \mathcal{P}_n(\mathbb{R})$ with $n \geq 1$ and $T \in \mathcal{L}(V)$ be the differentiation operator. Show that T does not have a square root, i.e. there does not exist $S \in \mathcal{L}(V)$ such that $S^2 = T$.

8. Let $T \in \mathcal{L}(V)$ be a linear operator and $U \subseteq V$ be a T-invariant subspace of V. Suppose $\mathbb{F} = \mathbb{C}$ and T has eigenvalues $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$. Define

$$U_i = U \cap G(\lambda_i, T)$$

for each $i \in \{1, \ldots, k\}$. Show that

$$U = \bigoplus_{i=1}^{k} U_i$$